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are the tangents at these points. The conic is, therefore, determined; for we have three points M, A, A' and the tangent at each of them, known.

From this point on the construction may be completed in any one of three ways:

(1) By the use of Pascal's Theorem, find the point H in which the normal at M cuts the conic again. Draw the circle on HM as diameter. Find the fourth point of intersection G of this circle with the conic. Draw HG. Through M draw MK parallel to HG and determine the point K in which this line cuts the conic again. The circle tangent to the conic at M and passing through K is the required circle of curvature at M. (Cf. L. Cremona, *Elements of Projective Geometry*, Oxford, 1885, p. 190).

(2) By the method previously described both the ellipse and the hyperbola that have center C, focus F, and point M are determined. The construction may, therefore, be completed by making use of the fact that at the intersection of two confocal conics the center of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

(3) Through F, draw a line parallel to the tangent at M. By Desargues' Theorem, find the points R and R' in which this line meets the conic. On MF take Q so that $MQ = RR'$. Then the circle through Q and tangent to the conic is the required circle of curvature; for, the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point. For results in (2) and (3) the reader may consult Salmon's *Conic Sections*, 6th ed., 1879, pp. 374–376.

It may be noted that if C is at infinity, the conic becomes a parabola with the axis CF, the focus F, and the point M given. Two parabolas may be drawn satisfying these conditions according to which of the bisectors of the angle made by MF and the diameter through M is taken as tangent. The above methods of construction then apply to this case also.

Also solved by H. HALPERIN, A. PELLETIER, J. B. REYNOLDS, and the Proposer.

2771 [1919, 191]. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy plane. Determine the portion of the area of the surface above the plane and between the xz and yz planes.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

A solution by vector analysis. Let $r = a\cos\theta + b\sin\theta$ be the polar equation of the circle revolved. From this we see that for a vector equation of the surface generated, we may write

$$r = (a\cos t + b\sin t)\cos t\cos u\cdot i + (a\cos t + b\sin t)\cos t\sin u\cdot j + (a\cos t + b\sin t)\sin t\cdot k;$$

or

$$r = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot j + \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\}\cdot k$$

from which

$$\frac{\partial r}{\partial t} = \{-a\sin 2t + b\cos 2t\}\cos u\cdot i + \{-a\sin 2t + b\cos 2t\}\sin u\cdot j + \{a\cos 2t + b\sin 2t\}\cdot k,$$

$$\frac{\partial r}{\partial u} = -\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot j,$$

giving the vector product

$$\begin{aligned}\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial u} = &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\cos u\cdot i\\ &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\sin u\cdot j\\ &+\tfrac{1}{2}\{-a\sin 2t + b\cos 2t\}\{a + a\cos 2t + b\sin^2 2t\}\cdot k.\end{aligned}$$

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[1] Of course it is unnecessary to describe the circle through F', F, M, for the tangent and normal bisect angles given at M; nor is it necessary to invoke an involution in order to get A and A' since we have given $F'M + FM = 2a$.—EDITOR.

Now since a differential element of the surface generated is the absolute value of $\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial u}$ multiplied by $dr\,du$, we have

$$S = \int\int \left[\frac{\partial r}{dt} \times \frac{\partial r}{\partial u}\right]_0 dudt$$

$$= \frac{\sqrt{a^2+b^2}}{2} \int\int \{a + a\cos 2t + b \sin 2t\} dtdu.$$

In order to get the limits of this integral, we must find the value of t for which the plane $r = xi + yj + y\tan\alpha\cdot k$, the plane through the x-axis making an angle α with the xy plane, cuts the surface of revolution. Equating coefficients of i, j, and k in the two equations, we find

$$x = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u, \qquad y = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u$$

$$y\tan\alpha = \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\},$$

the latter two of which give us the equation

$$b\cot\alpha - a\sin u + (a\cot\alpha - b\sin u)\sin 2t - (b\cot\alpha + a\sin u)\cos 2t = 0;$$

whence, $t = \tan^{-1}(-a/b)$ or $\tan^{-1}(\sin u/\cot\alpha)$ and since the first of these two values of t is for the special point where $r = 0$, we have

$$S = \frac{\sqrt{a^2+b^2}}{4}\int_0^{\pi/2} du\{2at + a\sin 2t - b\cos 2t\}_{\tan^{-1}\frac{\sin u}{\cot\alpha}}^{\pi/2}$$

$$= \frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2}\left\{\frac{\pi a}{2} - a\frac{\cot\alpha\sin u}{\csc^2\alpha - \cos^2 u} + b\frac{\cot^2\alpha\csc^2 u}{\csc^2\alpha + \cot^2\alpha\cot^2 u} - a\tan^{-1}\frac{\sin u}{\cot\alpha}\right\}du$$

$$= \frac{\sqrt{a^2+b^2}}{2}\left\{\frac{\pi^2 a}{4} + \frac{\pi b\cos\alpha}{2} + a\frac{\cos\alpha}{2}\log\left[\frac{1-\sin\alpha}{1+\sin\alpha}\right]\right\} - a\frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right)du$$

for the required surface. If $\alpha = \pi/2$, $S = 0$. If $a = 0$, $S = \pi b^2/4\cos\alpha$, one fourth the surface of a sphere of diameter b less half a zone of height $b/2(1 - \cos\alpha)$. If $\alpha = 0$ and $b = 0$, $S = \pi^2 a^2/8$ one eighth the surface of a torus. If $\alpha = 0$,

$$S = \frac{\pi^2 a\sqrt{a^2+b^2}}{8} + \frac{\pi b\sqrt{a^2+b^2}}{4},$$

as it should, since a semicircle of length $\frac{\pi}{2}\sqrt{a^2+b^2}$ is revolved through 90° about an axis at a distance $a/2 + b/\pi$ from its center of gravity. Since for $\alpha < 45°$, $\tan^{-1}(\sin u/\cot\alpha)$ will expand into an absolutely convergent series we may use for such values of α to any desired degree of accuracy

$$\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right)du = \sum_{n=1}^{n=\infty}(-1)^{n+1}(\tan\alpha)^{2n-1}\frac{2\cdot4\cdot6\cdots 2n-2}{3\cdot5\cdot7\cdots 2n-1}\frac{1}{2n-1}.$$

II. Solution by B. F. Finkel, Drury College.

Those of our readers who are unfamiliar with quaternions or in whose hands this subject is not an easily operated instrument of investigation can obtain the integral S by the following process:

Using the same notation; that is, taking r, t, and u as polar coördinates, we have for the equation of the surface of revolution $r = a\cos t + b\sin t$, and for the element of arc on circle revolved $ds = \sqrt{a^2+b^2}\cdot dt$.

The required area is expressed by the double integral

$$S = \int\int r\cos t\,du\,ds$$

$$= \sqrt{a^2+b^2}\int\int(a\cos^2 t + b\sin t\cos t)\,du\,dt.$$

The limits for u are 0 and $\pi/2$, and as the equation of the given plane is $\tan t = \tan \alpha \sin u$, the limits for t will be $\tan^{-1} (\tan \alpha \sin u)$ and $\pi/2$.

Therefore

$$S = \frac{\sqrt{a^2 + b^2}}{2} \int_0^{\pi/2} du \int_{\tan^{-1} (\tan \alpha \sin u)}^{\pi/2} (a + a \cos 2t + b \sin 2t)\, dt.$$

2772 [1919, 171]. Proposed by HARRY LANGMAN, New York City.

Given $1 = (-\frac{1}{2} + x)^r = (-\frac{1}{2} - x)^r$, where r is integral. Prove that r is a multiple of 3. In general, if

$$1 = \left[\cos \frac{2\pi}{m} + x\right]^r = \left[\cos \frac{2\pi}{m} - x\right]^r,$$

where r and m are integers, prove that r is a multiple of m.[1]

SOLUTION BY THE PROPOSER.

The general case only will be considered, and the trivial case $r = 0$ excluded.

Put

$$(1) \qquad a = \cos \frac{2\pi}{m} + x \qquad \text{and} \qquad b = \cos \frac{2\pi}{m} - x.$$

Then

$$(2) \qquad a + b = 2 \cos \frac{2\pi}{m}.$$

We may write

$$(3) \qquad 1 = \cos 2k\pi + i \sin 2k\pi.$$

Hence,

$$(4) \qquad a = \cos \frac{2k\pi}{r} + i \sin \frac{2k\pi}{r} \qquad \text{and} \qquad b = \cos \frac{2k'\pi}{r} + i \sin \frac{2k'\pi}{r},\ 0 < k,\ k' < r + 1.$$

From (2),

$$\cos \frac{2k\pi}{r} + \cos \frac{2k'\pi}{r} + i \left(\sin \frac{2k\pi}{r} + \sin \frac{2k'\pi}{r}\right) = 2 \cos \frac{2\pi}{m},$$

from which

$$(5) \qquad \cos (k + k') \frac{\pi}{r} \cdot \cos (k - k') \frac{\pi}{r} = \cos \frac{2\pi}{m} \qquad \text{and} \qquad \sin (k + k') \frac{\pi}{r} \cdot \cos (k - k') \frac{\pi}{r} = 0.$$

Hence, we must have

$$(6) \qquad \sin (k + k') \frac{\pi}{r} = 0.$$

From the problem, if $m \neq 1$, we must have $a \neq b$. Hence, in (4) $k \neq k'$. Hence, we must have $k + k' < 2r$, that is, $(k + k')/r < 2$. But, from (6), $(k + k')/r$ must be integral. Hence, since $k + k' > 0$,

$$(7) \qquad k + k' = r \quad \text{and} \quad k - k' = 2k - r.$$

From the first equation of (5), we then obtain (8) $\cos \frac{2\pi}{m} = -\cos \frac{2k - r}{r} \pi$.

Now $\frac{2}{m} \leqq 1$ and $\left|\frac{2k - r}{r}\right| \leqq 1$. Therefore, from (8),

$$\left|\frac{2k - r}{r}\right| = 1 - \frac{2}{m}.$$

Hence, $\frac{2k}{r} - 1 = 1 - \frac{2}{m}$ or $1 - \frac{2k}{r} = 1 - \frac{2}{m}$, from which $k = r - \frac{r}{m}$ or $k = \frac{r}{m}$. From (7), $k' = \frac{r}{m}$ or $k' = r - \frac{r}{m}$. Since k and k' are integral, we must have $\frac{r}{m}$ integral. Hence, r is a multiple of m.

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[1] If $m = 4$ we may take $x = 1$, $r = 2$; in this case r is not a multiple of m. Therefore the heorem of the question is not true for this case.—EDITORS.